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THE HARMONIC ANALYSIS OF THE SEMICIRCLE AND OF THE ELLIPSE

By A. E. Kennelly

If the diameter of the semicircle in Figure 1 is equal to π, the equation of the semicircle is $y = \sqrt{x(\pi - x)}$. Let it be required to express it in the form

$$y = \sum_{m=1}^{m=\infty} a_m \sin mx.$$

This calls for the development of $\sqrt{x(\pi - x)}$ into a Fourier sine-series, the development holding good from $x = 0$ to $x = \pi$. We have

$$a_m = \frac{2}{\pi}\int_0^{\pi} y \sin mx \, dx.$$

Express the integral in terms of the angle α, which may be considered

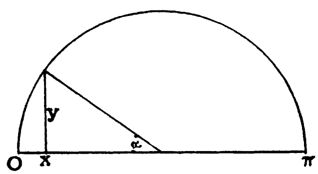

Fig. 1.

as increasing positively from 0 to π radians as x increases from 0 to π units.

Then
$$y = \frac{\pi}{2} \sin \alpha,$$

$$x = \frac{\pi}{2}(1 - \cos \alpha),$$

$$dx = \frac{\pi}{2} \sin \alpha \cdot d\alpha,$$

and the transformed integral is

$$a_m = \frac{2}{\pi}\int_0^\pi \frac{\pi}{2} \sin a \cdot \sin m \frac{\pi}{2}\left(1 - \cos a\right) \cdot \frac{\pi}{2} \sin a \cdot da$$

$$= \frac{\pi}{2}\int_0^\pi \sin^2 a \cdot \sin\left(\frac{m\pi}{2} - \frac{m\pi}{2} \cos a\right) da$$

$$= \frac{\pi}{2}\int_0^\pi \sin^2 a \left\{\sin \frac{m\pi}{2} \cos\left(\frac{m\pi}{2} \cos a\right) - \cos \frac{m\pi}{2} \sin \left(\frac{m\pi}{2} \cos a\right)\right\} da.$$

For any uneven integral value of m, this becomes:

$$\pm a_m = \frac{\pi}{2}\int_0^\pi \sin^2 a \cos\left(\frac{m\pi}{2} \cos a\right) da$$

$$= \frac{\pi}{2}\int_0^\pi \sin^2 a \left\{1 - \frac{\left(\frac{m\pi}{2}\right)^2 \cos^2 a}{2\,!} + \frac{\left(\frac{m\pi}{2}\right)^4 \cos^4 a}{4\,!} - \frac{\left(\frac{m\pi}{2}\right)^6 \cos^6 a}{6\,!} + \cdots\right\} da$$

$$= \frac{\pi}{2}\left[\int_0^\pi \sin^2 a\, da - \frac{\left(\frac{m\pi}{2}\right)^2}{2\,!}\int_0^\pi \sin^2 a \cos^2 a\, da + \frac{\left(\frac{m\pi}{2}\right)^4}{4\,!}\int_0^\pi \sin^2 a \cos^4 a\, da - \cdots\right]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2} - \frac{\left(\frac{m\pi}{2}\right)^2}{2\,!}\cdot\frac{\pi}{8} + \frac{\left(\frac{m\pi}{2}\right)^4}{4\,!}\cdot\frac{\pi}{16} - \frac{\left(\frac{m\pi}{2}\right)^6}{6\,!}\cdot\frac{5\pi}{128} + \frac{\left(\frac{m\pi}{2}\right)^8}{8\,!}\cdot\frac{7\pi}{256} - \cdots\right]$$

$$= \frac{\pi^2}{4}\left[1 - \frac{\left(\frac{m\pi}{2}\right)^2}{2\,!}\cdot\frac{1}{4} + \frac{\left(\frac{m\pi}{2}\right)^4}{4\,!}\cdot\frac{1}{8} - \frac{\left(\frac{m\pi}{2}\right)^6}{6\,!}\cdot\frac{5}{64} + \frac{\left(\frac{m\pi}{2}\right)^8}{8\,!}\cdot\frac{7}{128} - \cdots\right]$$

$$= \frac{\pi^2}{4}\left[1 - \left(\frac{m\pi}{4}\right)^2\cdot\frac{1}{2\,!} + \left(\frac{m\pi}{4}\right)^4\cdot\frac{2}{4\,!} - \left(\frac{m\pi}{4}\right)^6\cdot\frac{5}{6\,!} + \left(\frac{m\pi}{4}\right)^8\cdot\frac{14}{8\,!} - \cdots\right]$$

$$= \frac{\pi^2}{4}\left[\frac{1}{1!\,0!} - \left(\frac{m\pi}{4}\right)^2\cdot\frac{1}{2!\,1!} + \left(\frac{m\pi}{4}\right)^4\cdot\frac{1}{3!\,2!} - \left(\frac{m\pi}{4}\right)^6\cdot\frac{1}{4!\,3!} + \left(\frac{m\pi}{4}\right)^8\cdot\frac{1}{5!\,4!} - \cdots\right].$$

For any even integral value of m, $a_m = 0$. Consequently the equation of the semicircle is equivalent to the following:

$$y = \frac{\pi^2}{4}\left[\sin x \sum_1^\infty \frac{(-1)^{n+1}}{n!(n-1)!}\left(\frac{\pi}{4}\right)^{2(n-1)} - \sin 3x \sum_1^\infty \frac{(-1)^{n+1}}{n!(n-1)!}\left(\frac{3\pi}{4}\right)^{2(n-1)} + \cdots\right].$$

Each of these terms contains an infinite convergent series. Each term is affected with the positive sign.

Considering the first term within the bracket, the coefficient of sin x is

$$\frac{1}{1}-\frac{1}{2!}\left(\frac{\pi}{4}\right)^2+\frac{1}{3!\,2!}\left(\frac{\pi}{4}\right)^4-\frac{1}{4!\,3!}\left(\frac{\pi}{4}\right)^6+\cdots$$

or

$$1-a+\beta-\gamma+\delta-\cdots.$$

Then the corresponding series for the m^{th} harmonic is

$$1-m^2a+m^4\beta-m^6\gamma+m^8\delta-\cdots.$$

As m increases, more terms must be added in order to approach the final limit within an assigned error.

TABLE I

$\log_{10} a$	=	$\bar{1}$.	489	1497	671
β	=	$\bar{2}$.	501	1782	795
γ	=	$\bar{3}$.	212	1767	963
δ	=	$\bar{5}$.	701	3265	635
ϵ	=	$\bar{6}$.	014	3850	716
ζ	=	$\bar{8}$.	181	3155	440
η	=	$\overline{10}$.	223	3072	798
θ	=	$\overline{12}$.	156	1545	462
ι	=	$\overline{15}$.	992	0917	996
κ	=	$\overline{17}$.	740	8788	772
λ	=	$\overline{19}$.	410	4847	088
μ	=	$\overline{21}$.	007	5398	733
ν	=	$\overline{24}$.	537	6482	481
ξ	=	$\overline{26}$.	005	6087	161
o	=	$\overline{29}$.	415	5772	371
π	=	$\overline{32}$.	771	1880	958
ρ	=	$\overline{34}$.	075	6464	321

Table I gives the common logarithms of the quantities α, β, γ, etc., as far as ρ, the 18^{th} term in the series for the coefficient of $\sin x$.

The computed series for y as far as the fourth term is:

$$y = 1{\cdot}78073 \sin x + 0{\cdot}29494 \sin 3x + 0{\cdot}13274 \sin 5x + 0{\cdot}07903 \sin 7x + \cdots.$$

In order to attain a degree of accuracy sufficient to ensure the fifth decimal digit, six terms are necessary for $\sin x$, nine terms for $\sin 3x$, thirteen terms for $\sin 5x$, and eighteen terms for $\sin 7x$.

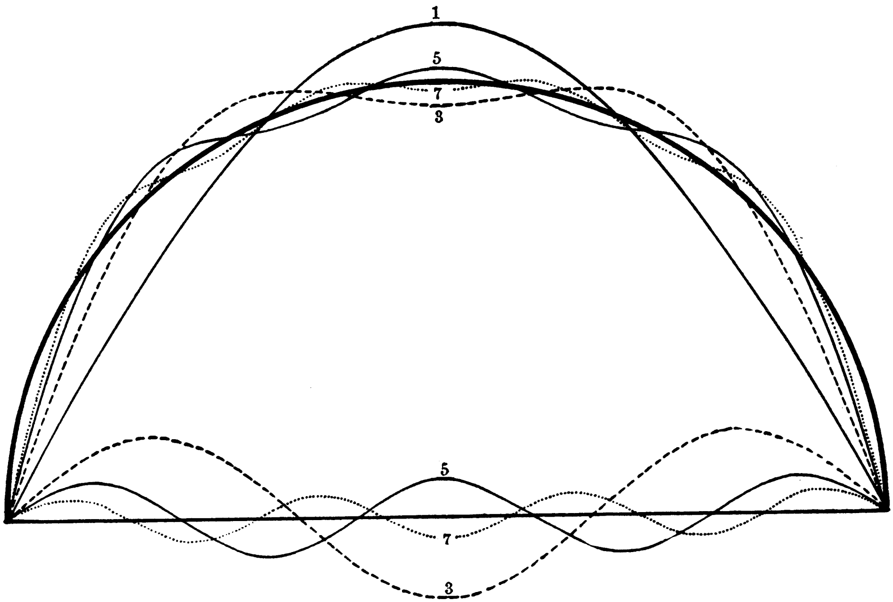

FIG. 2.

The accompanying diagram in Figure 2 shows the relations between the fundamental sinusoid and the sum of the first two, three, and four terms in the series. The line of points gives the final sum of all the sinusoids including the 7^{th}-frequency harmonic.

Analysis of the Ellipse.

In Figure 3, let $O\,a\,b\,\pi$ be an elliptic curve, the semicircle $O\,A\,B\,\pi$ being the upper half of the auxiliary circle. Let q be the projective ratio $\dfrac{a_1 a}{a_1 A}$

of the ellipse; then by the projective property of the ellipse, the component sinusoids will all be q times the corresponding sinusoids of the auxiliary circle; so that the series of harmonics for the ellipse is

$$y = q\frac{\pi^2}{4}\left[\sin x \sum_1^\infty \frac{(-1)^{n+1}}{n!(n-1)!}\left(\frac{\pi}{4}\right)^{2(n-1)} - \sin 3x \sum_1^\infty \frac{(-1)^{n+1}}{n!(n-1)!}\left(\frac{3\pi}{4}\right)^{2(n-1)} + \cdots\right].$$

Since the mean ordinate of a sinusoid is $\frac{2}{\pi}$ times the maximum, the area of a sinusoid is $\frac{2}{\pi}A \cdot B$; where A is the amplitude and B is the semi-wave length

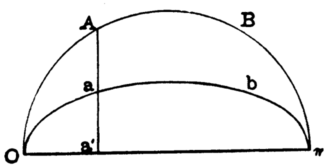

FIG. 3.

of the sinusoid. The area of a semicircle must be the same as the area of the harmonic components.

The area of the fundamental is $\frac{2}{\pi}\left\{\frac{\pi^2}{4}\sum_1^\infty \frac{(-1)^{n+1}}{n!(n-1)!}\left(\frac{\pi}{4}\right)^{2(n-1)}\right\}\pi,$

The area of the triple harmonic series is $\frac{2}{\pi}\left\{\frac{\pi^2}{4}\sum_1^\infty \frac{(-1)^{n+1}}{n!(n-1)!}\left(\frac{3\pi}{4}\right)^{2(n-1)}\right\}\frac{\pi}{3},$

The area of the quintuple harmonic series is $\frac{2}{\pi}\left\{\frac{\pi^2}{4}\sum_1^\infty \frac{(-1)^{n+1}}{n!(n-1)!}\left(\frac{5\pi}{4}\right)^{2(n-1)}\right\}\frac{\pi}{5},$

and so on. The total area will be

$$\frac{\pi^2}{2}\left\{\sum_1^\infty \frac{(-1)^{n+1}}{n!(n-1)!}\left(\frac{\pi}{4}\right)^{2(n-1)} - \frac{1}{3}\sum_1^\infty \frac{(-1)^{n+1}}{n!(n-1)!}\left(\frac{3\pi}{4}\right)^{2(n-1)} + \cdots\right\}.$$

This will be equal to the area of the semicircle or $\pi^3/8$. Consequently

$$\frac{\pi}{4}=\sum_1^\infty \frac{(-1)^{n+1}}{n!(n-1)!}\left(\frac{\pi}{4}\right)^{2(n-1)}-\frac{1}{3}\sum_1^\infty \frac{(-1)^{n+1}}{n!(n-1)!}\left(\frac{3\pi}{4}\right)^{2(n-1)}$$
$$+\frac{1}{5}\sum_1^\infty \frac{(-1)^{n+1}}{n!(n-1)!}\left(\frac{5\pi}{4}\right)^{2(n-1)}-\cdots,$$

which serves as an interesting check upon the correctness of the development. The values for these terms as far as the fourth are:

$$0\cdot721703 \quad + \quad \frac{0\cdot119536}{3} \quad + \quad \frac{0\cdot053798}{5} \quad + \quad \frac{0\cdot032031}{7} \quad + \cdots,$$

or $$\frac{\pi}{4} = 0\cdot721703 + 0\cdot039845 + 0\cdot010760 + 0\cdot004576 + \cdots$$
$$= 0\cdot776884 + \text{terms of higher order.}$$

The value of $\pi/4$ to six places being $0\cdot785398$, the sum of the remaining terms to infinity will be $0\cdot008514$.

If we introduce Bessel's functions of the first order, viz:

$$J_1(x) = \frac{x}{2}\left[1 - \frac{1}{1!\ 2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\ 3!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\ 4!}\left(\frac{x}{2}\right)^6 + \cdots\right],$$

we shall have

$$\sum_1^\infty \frac{(-1)^{n+1}}{n!(n-1)!}\ (x)^{2(n-1)} = \frac{1}{x}\ J_1(2x),$$

so that the expansion on page 50 may be written in the form

$$y = \pi\left[J_1\left(\frac{\pi}{2}\right)\ \sin x - \frac{1}{3}J_1\left(\frac{3\pi}{2}\right)\cdot \sin 3x + \frac{1}{5}\ J_1\left(\frac{5\pi}{2}\right)\cdot \sin 5x - \cdots\right],$$

and the above check becomes

$$\left(\frac{\pi}{4}\right)^2 = J_1\left(\frac{\pi}{2}\right) - \frac{1}{3^2}\ J_1\left(\frac{3\pi}{2}\right) + \frac{1}{5^2}\ J_1\left(\frac{5\pi}{2}\right) - \cdots.$$

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